Control of projective synchronization in chaotic systems

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We show that the scaling factor of projective synchronization in coupled partially linear systems is unpredictable. This gives rise to the difficulty in estimating the state of synchronized dynamics. We therefore propose a control method to manipulate the scaling factor onto any desired value so that the synchronization can be managed in a preferred way. A control law is derived based on the mechanism of projective synchronization of three-dimensional systems and an application is illustrated for the Lorenz system.

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Erratic motions governed by chaotic systems tend to move in a similar rhyme in terms of the angular phase or the amplitude through a certain coupling relationship. This phenomenon, referred to as chaos synchronization, was first observed by Pecora and Carroll [1] in 1990. This dynamical behavior has been widely explored in a variety of systems including physical [2–5], chemical [6–8], and ecological systems [9–11], to name just a few. Practical applications have emerged in secure communications [2,3] in which synchronization is used for signal processing.

In coupled chaotic systems, the dynamical state (e.g., amplitude) resulting from synchronization is usually unpredictable (we shall discuss this later). From an application point of view, it is necessary to control the system in order to offer the opportunity to select and manipulate the solution of synchronization in a defined way. In this Brief Report, we are interested in applying this idea to the so-called projective synchronization recently reported by Mainieri and Rehacek [12]. Projective synchronization is the dynamical behavior in which the responses of two identical systems synchronize up to a constant scaling factor. This phenomenon was observed in the coupled partially linear systems

$$\begin{aligned} \dot{\mathbf{u}}_m &= \mathbf{A}(z)\cdot\mathbf{u}_m, \\ \dot{z} &= f(\mathbf{u}_m, z), \\ \dot{\mathbf{u}}_s &= \mathbf{A}(z)\cdot\mathbf{u}_s. \end{aligned} \tag{1}$$

The state vector $\mathbf{u}$ has a linear form related to its derivative with respect to time. The matrix $\mathbf{A}(z)$ is only dependent on the variable z, which is nonlinearly related to the variables in $\mathbf{u}$. The subscript of m denotes the master system and s the slave system. In the coupled system (1), the master system governs the variable z and the variable z drives the slave system. Two systems share the variable z throughout all the time in dynamical evolution.

An example of the partially linear chaotic system is the Lorenz system [13]

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= (\mu - z)x - y, \\ \dot{z} &= xy - \rho z, \end{aligned} \tag{2}$$

where the parameters are set to $\sigma=10$, $\mu=60$, and $\rho=\frac{8}{3}$. The matrix $\mathbf{A}(z)$ is a 2×2 matrix $\left[\begin{smallmatrix} -\sigma & \sigma \\ (\mu-z) & -1 \end{smallmatrix}\right]$ containing the variable z that is a chaotic component. The state of the master system is $\mathbf{u}_m=(x_m, y_m)$, and the state of the slave system is $\mathbf{u}_s=(x_s, y_s)$. The third equation in Eqs. (2) is the coupling function linking the master system with the slave system.

In three-dimensional systems, such as the Lorenz system, Mainieri and Rehacek [12] observed that the amplitudes of two coupled systems tends to a fixed scaling factor $\overline{\alpha}$ such that

$$\lim_{t\to\infty}\|\alpha\cdot\mathbf{u}_m - \mathbf{u}_s\| = 0. \tag{3}$$

This factor becomes a characteristic of projective synchronization in partially linear chaotic systems. To cast a vivid impression on this phenomenon, we plot a synchronized dynamics of two coupled Lorenz systems in Fig. 1(a). From Fig. 1(b), we can see that the angular phases of the two coupled systems asymptotically approach each other. Before the two phases merge, the ratio of the amplitudes of the two systems keeps on changing [Fig. 1(c)]. When the two phases become identical, the phase synchronization happens. At this time, the ratio of the amplitudes of the two systems becomes the scaling factor.

We will show that the scaling factor is unpredictable. For a selected initial condition, one can hardly anticipate what value of the scaling factor will be delivered in projective synchronization. Consequently, the system performance is unmanageable. To explain this, we study the evolution of the angular phase and scaling factor in the cylindrical coordinates, from which we have

$$\begin{aligned} \dot{r} &= (x\dot{x} + y\dot{y})/r, \\ \dot{\theta} &= (x\dot{y} - y\dot{x})/r^2, \\ \dot{z} &= \dot{z}. \end{aligned} \tag{4}$$

Substituting Eqs. (1) into Eqs. (4), we get

$$\dot{\theta} = \{\cos\theta, \sin\theta\}[\mathbf{B}][\mathbf{A}]\begin{Bmatrix}\cos\theta \\ \sin\theta\end{Bmatrix} = g(z,\theta), \tag{5}$$

$$\frac{\dot{r}}{r} = \{\cos\theta, \sin\theta\}[\mathbf{A}]\begin{Bmatrix}\cos\theta \\ \sin\theta\end{Bmatrix}, \tag{6}$$

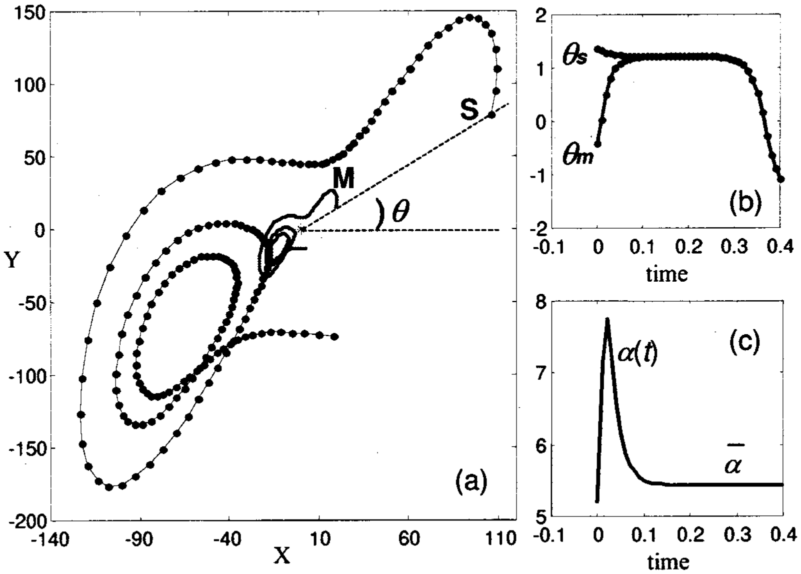

FIG. 1. In the projective synchronization (Lorenz), (a) the master (M) and slave (S) systems move in the same angular phase; (b) two angular phases evolve together; (c) the scaling factor tends to an unknown constant.

where $\mathbf{A}(z)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$. The components $a_{i,j}$ may associate with the variable z.

Mainieri and Rehacek [12] studied the evolution of the angular phases in projective synchronization. A difference $\varphi=\theta_m-\theta_s$ indicates a distance between the master and slave systems. From Eq. (5), the time variation of the angular difference is given by $\dot{\varphi}=g(z,\theta_m)-g(z,\theta_s)$, which carries a term of $\sin\varphi$. When $\sin\varphi$ tends to zero, it leads to $\dot{\varphi}\rightarrow 0$, which is a necessary condition of phase synchronization. The sufficient condition (stability condition) of the phase synchronization is given by the criterion $\int_0^t[\partial g(z,\theta)/\partial\theta]_{\theta=\theta_m}d\tau<0$, which is derived from a linear approximation of $\dot{\varphi}$ when $\sin\varphi\approx 0$ [12]. The necessary condition, i.e., $\sin\varphi=0$, implies that projective synchronization happens in two possible ways, i.e., $\varphi=0$ and $\varphi=\pi$, which are the two equilibrium points of $\dot{\varphi}$. As with $\varphi=0$, the state vectors of the master state and the slave state are synchronized in the same direction all the time, as displayed in Fig. 1. As with $\varphi=\pi$, the state vectors of two synchronized systems evolve in opposite directions, proportionally symmetrical to the origin.

To understand how the scaling factor finally forms, we look into what happens to the ratio of $\alpha=r_s/r_m$ when phase synchronization occurs. The time variation of this ratio is

$$\dot{\alpha}=\alpha\left(\frac{\dot{r}_s}{r_s}-\frac{\dot{r}_m}{r_m}\right). \tag{7}$$

Substituting Eq. (6) into Eq. (7), we have

$$\dot{\alpha}=\alpha h(z,\theta_m,\theta_s), \tag{8}$$

where

$$h(z,\theta_m,\theta_s)=[(a_{11}-a_{22})\sin(\theta_m+\theta_s) \\ -(a_{12}+a_{21})\cos(\theta_m+\theta_s)]\sin(\theta_m-\theta_s). \tag{9}$$

Integrating Eq. (8), we express the scaling factor as

$$\alpha(t)=\alpha(0)\exp\left(\int_0^t h(z,\theta_m,\theta_s)d\tau\right), \tag{10}$$

where $\alpha(0)$ is an initial scaling factor determined by the initial state. The function $h(z,\theta_m,\theta_s)$ in Eq. (9) also carries a term of $\sin\varphi$. It tends to zero as $\sin\varphi\rightarrow 0$. Thus the time integration in Eq. (10) reaches a limited value $\lim_{t\rightarrow\infty}\int_0^t h(z,\theta_m,\theta_s)d\tau=C$ when phase synchronization occurs. The value of the integration varies as $t<t_C$ and remains unchanged after $t\geqslant t_C$, where t_C is a critical time at which the phase synchronization happens. From Eq. (10), the final scaling factor is $\alpha(\infty)=\overline{\alpha}=\alpha(0)\times e^C$. Given an initial state [i.e., given $\alpha(0)$ that depends on the initial state of z, $\mathbf{u}_m$, and $\mathbf{u}_s$], the limit C becomes the only part to determine the scaling factor. Unfortunately, C is unpredictable because the variables in $h(z,\theta_m,\theta_s)$ behave chaotically. In literature [12], the authors remarked that a scaling factor depends on the initial conditions and a vector field, but it is difficult to obtain an analytical solution for the vector field. It implies that the prediction of a scaling factor is invalid, although the vector field seems a smoothly nonlinear function of the initial state (Fig. 5 in [12]). Our result [$\alpha(\infty)=\alpha(0)\times e^C$] delivers a similar conclusion that the scaling factor is determined by the initial conditions [i.e., $\alpha(0)$] and a nonlinear function [i.e., e^C, where $C=\lim_{t\rightarrow\infty}\int_0^t h(z,\theta_m,\theta_s)d\tau$], but the nonlinear function is hardly predicted because of the chaotic variables. To this end, we believe that any effort to estimate a resulting scaling factor in the initial stage works to no avail.

In this paper, we show that the scaling factor of projective synchronization can be arbitrarily maneuvered, as is required, by introducing a feedback control to the master system. A desired scaling factor for synchronization is essentially important since we can manipulate a synchronized dynamics to any scale by steering the master system. In what follows, we intend to develop a control method to make the projective synchronization manageable. Examining Eq. (10), we have realized that it delivers a useful message: an increase of the value of the integrand will lead to an exponential amplification of the scaling factor and vice versa. This message tells us that we can employ a control $\delta(t)$ to influence the result of synchronization. Thus we introduce a control to Eq. (8) and the system becomes

$$\dot{\alpha}=\alpha[h(z,\theta_m,\theta_s)+\delta(t)]. \tag{11}$$

However, it is still troublesome to operate the control function $\delta(t)$. First, the control contains only a small percentage of total energy of the system with the result that it may be deficient in strength and ability against $h(z,\theta_m,\theta_s)$ that varies chaotically. Second, operation of control in the time $t<t_C$ may spoil the natural evolution process of phase synchronization.

To overcome the problems, the strategy we have used is to take advantage of the fact that $h(z,\theta_m,\theta_s)$ becomes zero after $t\geqslant t_C$. In this time frame, we can exercise control without any problems. By so doing, we can see that the evolution of scaling factor (11) is solely controlled by $\delta(t)$ as

$$\dot{\alpha}=\alpha\delta(t), \quad t\geqslant t_C, \tag{12}$$

and at $t=t_C$, $\alpha(t_C)=\overline{\alpha}$ is the initial state of Eq. (12). The control function has a feedback form

$$\begin{aligned}\delta(t)&=0, \quad t<t_C,\\ \delta(t)&=\varepsilon(\alpha^*-\alpha), \quad t\geqslant t_C,\end{aligned} \tag{13}$$

where α^* is a desired scaling factor and the parameter ε is a feedback gain. From Eqs. (12) and (13), the controlled scaling factor is computed by

$$\alpha(t)=\alpha^*\left[1+\left(\frac{\alpha^*}{\overline{\alpha}}-1\right)\exp(-\varepsilon\alpha^*t)\right]^{-1}. \tag{14}$$

The control mechanism can be easily understood. From Eq. (13), the control function generates a negative signal to decrease the scaling factor (12) as $\alpha>\alpha^*$, and a positive signal to increase the scaling factor as $\alpha<\alpha^*$. The magnitude of the control signal decreases when the scaling factor approaches the target. The control (12) always leads to $\lim_{t\to\infty}\alpha(t)=\alpha^*$ at an exponential rate, if $\varepsilon\alpha^*>0$ as indicated in Eq. (14). Thus the control is robust and efficient. It is important to note that we must set $\varepsilon>0$ when $\alpha^*>0$ (synchronization at $\varphi=0$), and $\varepsilon<0$ when $\alpha^*<0$ (synchronization at $\varphi=\pi$). Also note that there exists singularity in Eq. (14) if the target value is set to zero, at which either the master system vanishes or the slave system tends to infinitely large. With this understanding, we now convert the control system (12) from the cylindrical coordinates to the Cartesian coordinates (x,y,z) only in which we can practice the control.

The control scheme is as follows. We only apply control to the master system, through the coupling variable z, to direct the slave system towards a desired state. During control, the phase synchronization persists all the time, but the scaling factor varies until it hits the target. We introduce the control functions ξ_x and ξ_y to the master system. The coupled systems now become

$$\begin{aligned}\dot{x}_m&=a_{11}x_m+a_{12}y_m+\xi_x,\\ \dot{y}_m&=a_{21}x_m+a_{22}y_m+\xi_y,\\ \dot{z}&=f(x_m,y_m,z),\\ \dot{x}_s&=a_{11}x_s+a_{12}y_s,\\ \dot{y}_s&=a_{21}x_s+a_{22}y_s,\end{aligned} \tag{15}$$

where $a_{i,j}$ may contain the variable z. Our next attempt is to derive the control functions ξ_x and ξ_y from Eqs. (12) and (13), which have been verified as being efficient and robust.

With the notation of $x=r\cos\theta$, $y=r\sin\theta$, and $\theta_m=\theta_s$ during phase synchronization, Eq. (12) can be written into two differential equations,

$$\begin{aligned}\frac{d}{dt}\left(\frac{x_s}{x_m}\right)&=\frac{x_s}{x_m}\varepsilon\left(\alpha^*-\frac{x_s}{x_m}\right),\\ \frac{d}{dt}\left(\frac{y_s}{y_m}\right)&=\frac{y_s}{y_m}\varepsilon\left(\alpha^*-\frac{y_s}{y_m}\right).\end{aligned} \tag{16}$$

We investigate the first equation in Eqs. (16) to derive the control function ξ_x. The derivation for ξ_y will be similar. The differentiation of the ratio of x_s/x_m with respect to time also yields

$$\frac{d}{dt}\left(\frac{x_s}{x_m}\right)=\frac{x_s}{x_m}\left(\frac{\dot{x}_s}{x_s}-\frac{\dot{x}_m}{x_m}\right). \tag{17}$$

From the first equation of Eqs. (16) and (17), we obtain

$$\frac{\dot{x}_s}{x_s}-\frac{\dot{x}_m}{x_m}=\varepsilon\left(\alpha^*-\frac{x_s}{x_m}\right). \tag{18}$$

Substituting the first equation and fourth equation of Eqs. (15) into Eq. (18), and noting that the relation $x_my_s=x_sy_m$ is held when phase synchronization persists, we then derive the control function for ξ_x,

$$\xi_x=\varepsilon(x_s-\alpha^*x_m). \tag{19}$$

Similarly, we derive the function for ξ_y,

$$\xi_y=\varepsilon(y_s-\alpha^*y_m). \tag{20}$$

The control should be turned on when $t\geqslant t_C$.

We now apply this control method to the Lorenz system (2) coupled in the form of (15). We illustrate the case of the synchronization at $\varphi=0$ in Fig. 2. Starting from the initial state $(\mathbf{u}_m,z,\mathbf{u}_s)=(0,1,10,1,1)$, the master and slave systems synchronize up to a scaling factor of 2.8426 without control. We want to manipulate the scaling factor to a higher value, say $\alpha^*=20$. This target can be reached as long as we set $\varepsilon>0$. By turning on the control $(\varepsilon=0.1)$, the size of the attractor of the slave system (dotted line) rapidly expands to 20 times as large as that of the master system. After the scaling

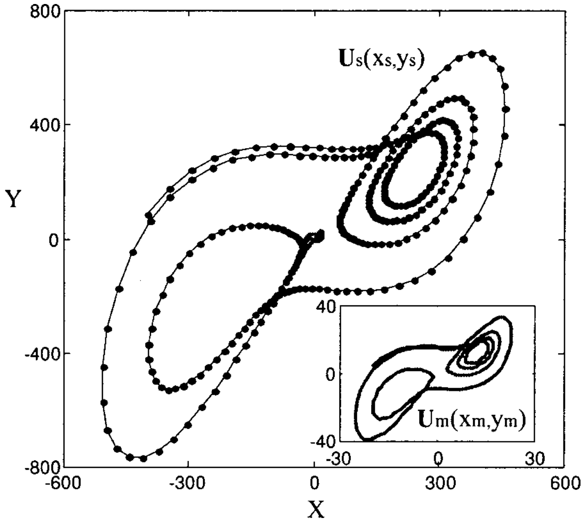

FIG. 2. The size of the chaotic attractor of the slave system (dotted line) is dramatically amplified to 20 times as large as that of the master system ($\alpha^*=20$ and $\varepsilon=0.1$).

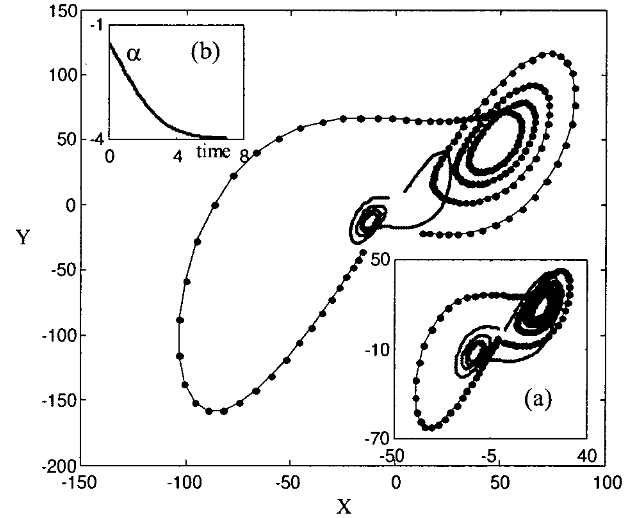

FIG. 3. The state vectors of the master (solid) and slave (dot) system move in the opposite directions proportionally symmetrical to the origin; (a) without control, $\overline{\alpha}=-1.5$; (b) the variation of α with control ($\alpha^*=-4$ and $\varepsilon=-0.2$).

factor is directed to the target, the control signal vanishes. The size of the master attractor almost remains the same before and after control, while the size of the slave attractor is dramatically amplified. The two chaotic attractors are similar in the structure but different in the size (see the inset). The state vectors of the two systems always remain in the same direction in synchronization.

Figure 3 illustrates the case of the synchronization at $\varphi=\pi$. Without control, the two systems synchronize up to a scaling factor of $\overline{\alpha}=-1.5$ as shown in Fig. 3(a). With control ($\varepsilon=-0.2$), the scaling factor is directed to $\alpha^*=-4$, plotted in Fig. 3(b). The slave system (dot) is pushed away from the master system (solid) displayed in Fig. 3. In general, this control approach works very well. In the case in which we manipulate the scaling factor from positive to negative or vice versa, the response of the master system may temporally vanish (i.e., $r_m=0$) when the sign of the scaling factor is changed. When the response r_m shrinks to the origin, the scaling factor may become very large but the control is still valid.

In summary, we proposed a control method of manipulation of the scaling factor to cope with the problem of the unpredictable performance in projective synchronization of partially linear chaotic systems. We have verified the efficiency and robustness of this approach theoretically and numerically. This control approach allows us to arbitrarily select and manipulate the outcome of synchronized dynamics in partially linear chaotic systems.

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